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The Linear Stability of No-Slip Boundary Conditions in the Numerical Solution of Nonsteady Fluid Flows

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This paper investigates the linear stability of finite-difference approximations to no-slip boundary conditions in fluid flow. Stability analyses are competed for one-dimensional problems using schemes suitable for two-dimensional problems. All conditions are shown to be stable if there is no suction and conditionally stable if there is suction. It is also shown that there are difficulties with choosing a suitable time integration method if a high order no-slip approximation is used.

1. INTRODUCTION

1.1 In this paper we investigate the effects of no-slip boundary conditions on the linear stability of difference equations arising in the numerical solution of the two-dimensional, time-dependent, Navier–Stokes equations. The usual method of investigating linear stability when diffusion teams are present consists of examining the behaviour of amplitudes of Fourier components assuming that the equations are approximately linear, and that the region is infinite (or that boundaries are periodic). This clearly does not take into account the effect of no-slip boundary conditions. The method employed here is equivalent to seeking Fourier components satisfying the boundary conditions. We allow these components to be complex.

The results obtained are for one-dimensional problems using finite-difference schemes applicable to two dimensions. Of course, in practice, more efficient methods could be used to solve the one-dimensional flow equations. We can expect that the stability properties of a two-dimensional problem will be similar to those of a one-dimensional problem with similar boundary conditions. Certainly, if a boundary condition causes instability in one dimension, it is most probable that there will be difficulties in a higher number of dimensions.

We shall also consider the effect of suction at a wall, when the linear parts of non-linear terms are introduced. We shall not discuss the moving wall and extrapolation conditions suggested by Fromm [1, 2]. An investigation of their stability is given by Taylor [3].

1.2. *Numerical Method*

The two-dimensional, time-dependent, Navier–Stokes equations may be written in terms of the stream function $\psi(x, y, t)$ and vorticity $\zeta(x, y, t)$ as

$$\frac{\partial \zeta}{\partial t} = \nu \nabla^2 \zeta + J(\psi, \zeta) \tag{1}$$

and

$$\nabla^2 \psi = -\zeta, \tag{2}$$

where ν is the viscosity,

$$J(\psi, \zeta) \equiv \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x},$$

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The velocity of the fluid at any point is given by

$$u = \frac{\partial \psi}{\partial y} \tag{3}$$

and

$$v = -\frac{\partial \psi}{\partial x}, \tag{4}$$

where u and v are the velocity components in the x and y directions.

The numerical method consists of replacing (1) and (2) by suitable finite-difference equations (see, e.g., [1]). Vorticity values are first advanced over a time step using a difference approximation to (1). The vorticity values so obtained are used in a difference approximation to (2) and the resulting linear equations are solved to find values for the stream function. Various difference schemes have been tried and particular attention has been directed to finding satisfactory difference analogues of the nonlinear term $J(\psi, \zeta)$. We shall assume that the operator ∇^2 in both (1) and (2) is replaced by the usual five-point difference formula so that (2) becomes

$$\psi_{i-1,j} + \psi_{i+1,j} + \psi_{i,j-1} + \psi_{i,j+1} - 4\psi_{i,j} = -h^2 \zeta_{i,j}, \tag{5}$$

581/6/2-8

where $\psi_{i,j}$ and $\zeta_{i,j}$ are the values of the stream function and vorticity in the difference scheme at the point $x_i = x_0 + ih$, $y_j = y_0 + jh$. Thus the points included in the scheme are on a square grid of mesh size h.

2. No-Slip Boundary Conditions

2.1. We shall consider numerical approximations to conditions on a fixed wall along the line $y = y_0$. We write ψ_0 and ζ_0 (instead of $\psi_{i,0}$ and $\zeta_{i,0}$) for the values of the stream function and vorticity at an arbitrary mesh point on this boundary, ψ_{-1} and ζ_{-1} for values one step outside the boundary, and ψ_1 and ζ_1 for values one step inside the boundary.

If there is no-slip at the wall, the stream function is known. Vorticity is generated at such a boundary and this must be incorporated in the numerical method. The process, proposed and used by Fromm, is to first advance vorticity values at *interior* points using a finite-difference form of (1). Values of ψ at interior points may now be found using Eq. (5), as these involve the vorticity at only interior points and ψ is known on the boundary. Using the condition of zero velocity along the wall, it is possible to obtain hypothetical values for ψ at points just outside the boundary. The values of ψ thus found may now be inserted in Eq. (5) for boundary points, to obtain the vorticity at these points. In the methods to be described, the stream function at the wall is a linear function of x and therefore there is a suction velocity which is constant along the wall.

2.2. *Fromm's Condition*

The conditions on the stream function are:

$$\psi_0 - \psi_{-1} = 0 \qquad \text{and} \qquad \psi_0 = \psi_b,$$

where ψ_b is a given linear function of x. Values of ψ at interior points are found using (5) with $\psi_0 = \psi_b$ at the boundary. Reversing (5), we obtain for the boundary vorticity,

$$\zeta_0 = -(\psi_1 - 2\psi_0 + \psi_{-1})/h^2$$

and thus, as $\psi_{-1} = \psi_0 = \psi_b$, we obtain

$$\zeta_0 = -(\psi_1 - \psi_b)/h^2. \tag{6}$$

Fromm [1] has shown that the condition is a more accurate approximation to flow with a wall along $y = y_0 - \frac{1}{2}h$ than with a wall along $y = y_0$.

2.3. *Thom's condition*

Thom [4] proposed the conditions:

$$\psi_1 - \psi_{-1} = 0 \qquad \text{and} \qquad \psi_0 = \psi_b,$$

where ψ_b is a given linear function of x.

The method is similar to Fromm's and the boundary vorticity is found using

$$\zeta_0 = -2(\psi_1 - \psi_b)/h^2. \tag{7}$$

This condition is often easier to apply than Fromm's.

2.4. *Wood's condition*

Woods [5] derived the conditions:

$$\psi_0 = \psi_b$$

and

$$\zeta_0 = -\frac{3}{h^2}(\psi_1 - \psi_b) - \tfrac{1}{2}\zeta_1, \tag{8}$$

where ψ_b is a given linear function of x.

This method uses an approximation to Eq. (2) of higher degree than (5) when finding the boundary vorticity values. It has been employed successfully in several calculations of steady-state solutions, e.g., Russell [6].

3. Numerical Stability

3.1. We shall investigate the linear stability of the usual difference approximation to the one-space-dimensional case of Eqs. (1) and (2). For one-dimensional flow along a wall y = a constant, (1) becomes

$$\frac{\partial\zeta(y,t)}{\partial t} = \nu\frac{\partial^2\zeta(y,t)}{\partial y^2} - V\frac{\partial\zeta(y,t)}{\partial y} \tag{9}$$

where $-V$ is the suction velocity at the wall. If we replace only the space derivatives of $\zeta(y,t)$ by finite differences, except at the boundaries, we obtain

$$\frac{d\zeta_j(t)}{dt} = \frac{\nu}{h^2}(\zeta_{j+1}(t) - 2\zeta_j(t) + \zeta_{j-1}(t)) - \frac{V}{2h}(\zeta_{j+1}(t) - \zeta_{j-1}(t)) \tag{10}$$

where $\zeta_j(t)$ denotes the approximation to $\zeta(y_j, t)$. All the difference approximations to $J(\psi, \zeta)$ described by Arakawa [7] and Fromm [8] reduce to the transportation term in (10). In any numerical method we would also have to make difference approximations to time derivatives.

The system (10) may be written in the form

$$\frac{d\boldsymbol{\zeta}}{dt} = \mathbf{M}\boldsymbol{\zeta} + \mathbf{b}, \tag{11}$$

where $\boldsymbol{\zeta}$ is a vector with components $\zeta_j(t)$, $\mathbf{M}$ is a square matrix, and $\mathbf{b}$ is a vector whose elements depend on the boundary conditions.

If λ_r, $r = 1, 2, \ldots, m$, are the eigenvalues of $\mathbf{M}$, we say that the system (11) is

stable, if $\max_r \operatorname{Re}(\lambda_r) < 0$,

neutrally stable, if $\max_r \operatorname{Re}(\lambda_r) = 0$ and every λ_r with $\operatorname{Re}(\lambda_r) = 0$ is a simple zero of the minimal polynomial of $\mathbf{M}$,

unstable, if it is neither stable nor neutrally stable.

In general, a perturbation of the initial solution vector of (15) will produce an error decreasing to zero with time for the stable case, a uniformly bounded error for the neutrally stable case and an unbounded error for the unstable case. See, e.g., Bellman [9].

Conditions on the eigenvalues of $\mathbf{M}$ to ensure stability of the complete discretization of (11) are fairly easily devised for the usual time-integration methods. Several methods are considered by Varga [10]. For all methods it is essential that (11) be stable or neutrally stable.

If $\mathbf{x}$ is an eigenvector of $\mathbf{M}$, corresponding to an eigenvalue λ_r, the components x_j of $\mathbf{x}$ satisfy the difference equations

$$(\alpha + \beta)\, x_{j-1} - (2\alpha + \lambda_r)\, x_j + (\alpha - \beta)\, x_{j+1} = 0 \tag{12}$$

where $\alpha = \nu/h^2$ and $\beta = V/(2h)$. Homogeneous boundary conditions on the x_j are derived from those applied to ζ_j.

The general solution of (12) (for $\theta_r \neq 0$ or π) is given by

$$x_j = a^j\,(A \cos j\theta_r + B \sin j\theta_r), \tag{13}$$

where

$$\lambda_r = 2\alpha\left(\frac{\cos\theta_r}{\operatorname{ch} b} - 1\right), \tag{14}$$

$$a = \frac{(\alpha + \beta)^{1/2}}{(\alpha - \beta)^{1/2}}, \tag{15}$$

and

$$b = \ln a. \tag{16}$$

We seek values of θ_r, A, and B which give nonzero eigenvectors such that the components (13) satisfy the boundary conditions introduced earlier. If a solution is given by $\theta_r = 0$ or π, the general solution (13) is no longer valid and we seek a solution containing terms of the form $j(\pm a)^j$. The following three cases regarding α and β will be considered and some numerical results are shown in Table I. (The case $\alpha = -\beta$ is ignored as for this (12) reduces to a first order equation.)

TABLE I

Values of $\max_r \mathrm{Re}(\lambda_r)$[a]

Boundary conditions		Case	B	C	C
$y = y_0$	$y = y_{n-1}$	β	$+0.5$	$+5.0$	$+15.0$
ψ_0 given	Fromm's		-0.35205 -0.30628	-1.16549 -1.34923	$+1.22316$ $+0.66670$
U given	Fromm's		-0.35612 -0.30726	-0.95308 -1.20822	$+2.41150$ $+1.62210$
ψ_0 given	Thom's		-0.35777 -0.30810	-0.39633 -0.75668	$+4.09456$ $+2.97769$
U given	Thom's		-0.36270 -0.30926	-0.02129 -0.51676	$+6.04077$ $+4.45470$
ψ_0 given	Woods'		-0.36021 -0.30881	$+5.62269$ $+5.89842$	$+26.6770$ $+27.5973$
U given	Woods'		-0.36530 -0.30997	$+6.50898$ $+6.51089$	$+29.6448$ $+29.6495$

[a] In all cases $\alpha = 1$ and for each set of boundary conditions and values of β the two entries are for $n = 10$ and 15 respectively.

Case A $\alpha > 0, \beta = 0$. This corresponds to $V = 0$. We obtain from (15) and (16) $a = 1$ and $b = 0$.

Case B $\alpha > \beta > 0$. This corresponds to $0 < V < 2\nu/h$. Both a and b are real and $a > 1$. We have

$$\mathrm{Re}(\lambda_r) = 2\alpha \left(\frac{\mathrm{Re}(\cos\theta_r)}{\mathrm{ch}\, b} - 1\right). \tag{17}$$

Case C $\beta > \alpha > 0$. This corresponds to $V > 2\nu/h$. We let $a = -ik$ where $k > 1$, when $\operatorname{ch} b = -i \operatorname{sh} c$, where $c = \ln k > 0$. From (14) we find

$$\lambda_r = 2\alpha\left(\frac{i\cos\theta_r}{\operatorname{sh} c} - 1\right) \tag{18}$$

and

$$\operatorname{Re}(\lambda_r) = -2\alpha\left(\frac{\operatorname{Im}(\cos\theta_r)}{\operatorname{sh} c} + 1\right). \tag{19}$$

4. Stability of Fromm's Condition

4.1. *Channel of Finite Width*

We consider first the stability of Fromm's condition when used as *both* walls of a channel in which the flow is one-dimensional with no suction at the walls. The boundary conditions, which are of the form (6), involve stream function values and these must be related to the vorticity if we are to obtain equations of the form (11). For the one-dimensional case, the difference approximation to (2) may be written as

$$\mathbf{H}\boldsymbol{\psi} = -h^2\boldsymbol{\zeta} = \mathbf{c}, \tag{20}$$

where the components of $\boldsymbol{\psi}$ and $\boldsymbol{\zeta}$ are the values of the stream function and vorticity respectively at *interior* grid points, $\mathbf{H}$ is the $n \times n$ tridiagonal matrix

$$\mathbf{H} = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & 1 & -2 & 1 \\ 0 & & 1 & -2 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} \psi_0 \\ 0 \\ \vdots \\ 0 \\ \psi_{n+1} \end{bmatrix}.$$

ψ_0 and ψ_{n+1} are the specified boundary values of the stream function. $y = y_0$ and $y = y_{n+1}$ are the walls of the channel. Now $\mathbf{H}^{-1}$ is symmetric and has i, jth element

$$-\frac{(n+1-j)\,i}{n+1} \text{ for } j \geqslant i,$$

and therefore,

$$\psi_1 = h^2 \sum_{j=1}^{n} \left(\frac{n+1-j}{n+1}\right) \zeta_j + \frac{n\psi_0 + \psi_{n+1}}{n+1}. \tag{21}$$

The boundary condition given by Eq. (6) becomes, for the lower boundary,

$$\zeta_0 = -\sum_{j=1}^{n} \left(\frac{n+1-j}{n+1}\right) \zeta_j + \frac{\psi_0 - \psi_{n+1}}{(n+1)\,h^2}. \tag{22}$$

The corresponding condition on the eigenvectors of **M** is, therefore,

$$x_0 = -\sum_{j=1}^{n} \left(\frac{n+1-j}{n+1}\right) x_j$$

i.e.,

$$\sum_{j=0}^{n} (n+1-j)\,x_j = 0. \tag{23}$$

Similarly at the upper boundary, we obtain

$$\sum_{j=1}^{n+1} jx_j = 0. \tag{24}$$

As there is no suction at the walls, $V = 0$ and we are interested in Case A when $a = 1$. On substituting the general solution (13) in (23) and (24) we obtain

$$A \sum_{j=0}^{n} (n+1-j) \cos j\theta_r + B \sum_{j=0}^{n} (n+1-j) \sin j\theta_r = 0 \tag{25}$$

and

$$A \sum_{j=1}^{n+1} j \cos j\theta_r + B \sum_{j=1}^{n+1} j \sin j\theta_r = 0. \tag{26}$$

These are two linear equations in the constants A and B. They yield a nonzero solution if the determinant of the coefficients of A and B is zero. After some considerable manipulation using identities for sums of finite trigonometric series (see, e.g., Taylor [3]) we can show that this occurs when the θ_r are roots of

$$f(\theta)/(1 - \cos\theta)^2 = 0,$$

where

$$f(\theta) \equiv \sin\theta[1 - \cos(n+2)\,\theta] - (n+2)[1 - \cos\theta] \sin(n+2)\,\theta.$$

Now

$$\begin{aligned} f\left(\frac{r\pi}{n+2}\right) &= \sin\left(\frac{r\pi}{n+2}\right) \cdot [1 - (-1)^r] \qquad \text{for } r = 1, 2, \dots, n+1, \\ &= 0 \qquad \text{for even } r, \\ &> 0 \qquad \text{for odd } r, \end{aligned}$$

and

$$f(0) = f(\pi) = 0.$$

Also we have

$$f'\left(\frac{r\pi}{n+2}\right) < 0 \qquad \text{for even } r,$$

and

$$f'(\pi) < 0 \qquad \text{for even } n.$$

We conclude that there are n real roots θ_r and the corresponding eigenvalues are given by (14) with ch $b = 1$. Each λ_r is real and negative and, therefore, (11) is stable.

4.2. *Semi-Infinite Region with Specified Stream Function at Infinity*

We consider now one-dimensional flow in the region $y_0 \leqslant y \leqslant y_{n+1}$, along a wall $y = y_{n+1}$,[1] with uniform flow assumed on a line, $y = y_0$, some distance from the wall. Along $y = y_0$, the vorticity satisfies $\zeta_0 = 0$ and the stream function ψ_0 is specified. This is the simplest type of approximation which may be made to flow at "infinity." It also applies along a line about which the flow is symmetric.

If ψ_0 is constant along $y = y_0$, the boundary is a streamline. If ψ_0 is a linear function of x, there is a steady flow across the boundary.

The stream function values are again given by (20). (Note that the difference $\psi_{n+1} - \psi_0$ determines the mass flow across any line joining the two boundaries.) The condition

$$\zeta_0 = 0$$

gives the condition

$$x_0 = 0$$

for the eigenvector components, and thus from (13), $A = 0$. At the upper boundary the x_j satisfy (24). We again sum the finite trigonometric series involved and deduce that we require the θ_r to be roots of

$$\begin{aligned}\{a(1 - a^2)\sin\theta - (n+2)\,a^{n+4}\sin n\theta + [2(n+2) + (n+1)\,a^2]\,a^{n+3}\sin(n+1)\,\theta \\ - [2(n+1)\,a^2 + (n+2)]\,a^{n+2}\sin(n+2)\,\theta \\ + (n+1)\,a^{n+3}\sin(n+3)\,\theta\}/(1 + a^2 - 2a\cos\theta)^2 = 0 \qquad (27)\end{aligned}$$

[1] We take the wall as $y = y_{n+1}$ as this simplifies the algebra needed in this section.

if B is to be nonzero. The roots of this equation are not real unless $a = 1$ and therefore full analysis has only been found possible for Case A. However it has been possible to complete a partial analysis for other cases and obtain conditions under which the method is unstable. We shall complete this in detail for this set of boundary conditions so as to illustrate the method.

Case A. $a = 1$ and by considering the behavior of the numerator in (27) at points $r\pi/(n + \frac{3}{2})$ we can show that there are n real roots θ_r.

Case B. The eigenvalues are complex and analysis has not been possible. In all of several particular examples investigated by direct calculation of eigenvalues, the case has been found stable. Table I gives some results.

Case C. This case corresponds to flow towards the wall $y = y_{n+1}$ at which there is suction. If n is odd, there must be a real eigenvalue of $\mathbf{M}$ as complex eigenvalues occur in conjugate pairs. We shall examine the sign of this real eigenvalue and show that, if β/α is sufficiently large, the method is unstable. This restriction to odd values of n is justified, in that we are concerned with seeking indications of the stability of the method in two dimensions.

If λ_r is real, we deduce from (18) that $\cos\theta_r$ is purely imaginary and we therefore let

$$\theta_r = \frac{\pi}{2} + i\phi_r,$$

when (18) becomes

$$\lambda_r = 2\alpha(\operatorname{sh}\phi_r/\operatorname{sh} c - 1).$$

We have $\operatorname{Re}(\lambda_r) > 0$ if ϕ_r is real and $\phi_r > c > 0$. Putting $\theta = \frac{1}{2}\pi + i\phi$ in (27) we deduce (remembering n is odd) that ϕ_r are the roots of

$$F(\phi)/(\operatorname{sh} c - \operatorname{sh}\phi)^2 = 0 \tag{28}$$

where

$$\begin{aligned} F(\phi) = {} & k(1 + k^2)\operatorname{ch}\phi - (n + 2)\,k^{n+4}\operatorname{ch} n\phi \\ & + [2(n + 2) - (n + 1)\,k^2]\,k^{n+3}\operatorname{sh}(n + 1)\,\phi \\ & + [2(n + 1)\,k^2 - (n + 2)]\,k^{n+2}\operatorname{ch}(n + 2)\,\phi - (n + 1)\,k^{n+3}\operatorname{sh}(n + 3)\,\phi, \end{aligned}$$

$a = -ik$ and $c = \ln k$.

Notice first that $F(c) = F'(c) = 0$, but this does not give a root $\phi = c$ of (28), as the denominator also has a double zero at this point.

If $F''(c)$ is positive, $F(\phi)$ has a minimum at $\phi = c$ and since

$$F(\phi) \sim -\frac{(n+1)}{2} k^{n+3} e^{(n+3)\phi} \qquad \text{as } \phi \to \infty,$$
$$< 0,$$

there will be a real root ϕ_r with $\phi_r > c$. Now, $F''(c) > 0$, if $k < k_c$, where k_c is the only root of

$$\tfrac{1}{2}(n+1)(n+2) + k^2(n^2+3n+3) + \tfrac{1}{2}k^4(n+1)(n+2) - (n+2)\,k^{2n+4} - (n+1)\,k^{2n+6} = 0 \tag{29}$$

with $k_c > 1$.

Thus if n is odd and $k < k_c$, the method is unstable.

Since

$$k^2 = -\frac{\alpha+\beta}{\alpha-\beta}$$

the condition $k < k_c$ is equivalent to

$$\frac{Vh}{2\nu} = \frac{\beta}{\alpha} > \frac{(k_c^2+1)}{(k_c^2-1)} = \left(\frac{\beta}{\alpha}\right)_c \qquad \text{(say).} \tag{30}$$

A graph of $(\beta/\alpha)_c$ against n, for odd n, is shown in Fig. 1. By computing the eigenvalues of **M** it has been found, in all of several numerical cases with n odd, that the real eigenvalue derived from (28) is the eigenvalue with largest real part (e.g., Table II gives eigenvalues of **M** for $n = 15$, $\alpha = 1$, $\beta = 11$). Thus, for odd n, (11) has been proved unstable for $(\beta/\alpha) > (\beta/\alpha)_c$ and on the basis of numerical

TABLE II

Eigenvalues of **M** for $\alpha = 1$, $\beta = 11$, $n = 15$, ζ_0 and ψ_0 Specified and Fromm's Condition for $y = y_{n+1}$

−0.15780
−0.49459 ± 4.40293i
−1.17858 ± 8.16412i
−1.42657 ± 11.4562i
−1.65094 ± 14.3632i
−1.79940 ± 16.7140i
−1.91326 ± 18.4651i
−1.97781 ± 19.5373i

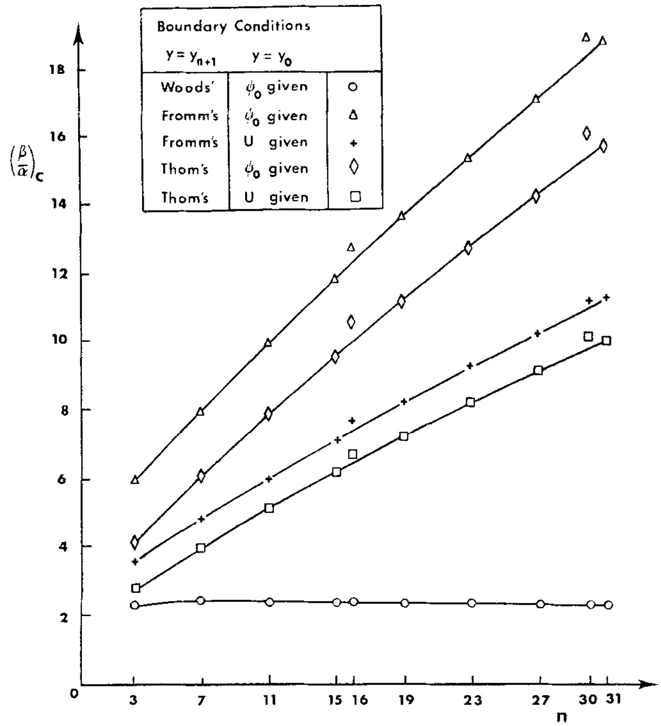

FIG. 1. Critical values of (β/α). For odd n the values were obtained analytically, e.g., from Eq. (33). For $n = 16,30$ the values were obtained by direct calculation of eigenvalues.

results it is probably stable for $(\beta/\alpha) < (\beta/\alpha)_c$. If n is even, direct calculation of the eigenvalues of **M** suggests that (11) becomes unstable, as (β/α) is increased, at points just above a smooth curve (Fig. 1) joining the critical values $(\beta/\alpha)_c$ for odd n. Critical values of (β/α) are shown in Fig. 1 for $n = 16, 30$. Approximately, therefore, the region above the curve represents unstable ratios β/α and the region below the curve, stable ratios.

4.3. *Semi-Infinite Region with Specified Velocity at Infinity*

In the last section we assumed that the stream function was specified at both boundaries, which would imply a fixed mass flow across any line joining the two boundaries. In a two-dimensional flow, it would probably be more appropriate to specify only the velocity along the boundary at infinity, so that stream function values may be adjusted according to other factors influencing the flow.

We shall consider the one-dimensional stability of flow in the region $y_0 \leqslant y \leqslant y_{n+1}$ along a wall $y = y_{n+1}$, with $y = y_0$ assumed to be a boundary at "infinity." Along $y = y_0$, we take $\zeta_0 = 0$ and $\psi_0 - \psi_{-1} = Uh$. The latter is a finite-difference approximation to

$$\frac{\partial \psi}{\partial y} = U,$$

where U is the specified velocity along $y = y_0$.

The stream function will be related to the vorticity by

$$\mathbf{L}\boldsymbol{\psi} = -h^2\boldsymbol{\zeta} + \mathbf{d},$$

where the components of $\boldsymbol{\psi}$ and $\boldsymbol{\zeta}$ are the stream function and vorticity at *interior* grid points, $\mathbf{L}$ is the $n \times n$ tridiagonal matrix,

$$\mathbf{L} = \begin{bmatrix} -1 & 1 & & 0 \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & 1 & -2 & 1 \\ 0 & & 1 & -2 \end{bmatrix} \text{ and } \mathbf{d} = \begin{bmatrix} Uh \\ 0 \\ \vdots \\ 0 \\ -\psi_{n+1} \end{bmatrix},$$

where ψ_{n+1} is the specified stream function at the wall. The last row of $\mathbf{L}^{-1}$ has -1 for each element and thus

$$\psi_n = h^2 \sum_{j=1}^{n} \zeta_j - Uh + \psi_{n+1} \tag{31}$$

and, on using Eq. (6) modified for an upper boundary, we obtain

$$\zeta_{n+1} = -\sum_{j=1}^{n} \zeta_j + \frac{U}{h}. \tag{32}$$

The boundary conditions on the eigenvalues of $\mathbf{M}$ are thus $x_0 = 0$ and

$$\sum_{j=1}^{n+1} x_j = 0.$$

The first condition implies $A = 0$ and from the second we deduce that the θ_r must be roots of

$$[\sin\theta + a^{n+2}\sin(n+1)\,\theta - a^{n+1}\sin(n+2)\,\theta]/(\operatorname{ch} b - \cos\theta) = 0. \tag{33}$$

Cases A and B. By considering the numerator in (33) for points $r\pi/(n + \frac{3}{2})$, it is easy to show there are n real roots and thus (11) is stable.

Case C. Analysis similar to that of the Section 4.2 may be made for odd n. Figure 1 shows critical values of β/α and some numerical results are shown in Table I.

5. Stability of Thom's Condition

5.1. The treatment and results for Thom's condition are similar to those of Fromm's. If ψ_0 and ψ_{n+1} are specified the vorticity is given by

$$\zeta_{n+1} = -2 \sum_{j=1}^{n} \left(\frac{j}{n+1}\right) \zeta_j + \frac{2(\psi_{n+1} - \psi_0)}{(n+1)\, h^2}$$

at the upper boundary. If only the velocity is specified at the lower boundary we obtain

$$\zeta_{n+1} = -2 \sum_{j=1}^{n} \zeta_j + \frac{2U}{h}.$$

For one-dimensional flow in a channel of finite width the condition gives a stable system of ordinary differential equations. For a semi-infinite region, critical values of β/α are shown in Fig. 1.

6. Stability of Woods' Condition

6.1. *Semi-Infinite Region with Specified Stream Function at Infinity*

If Woods' condition is used along $y = y_{n+1}$ and the conditions $\zeta_0 = 0$ and specified ψ_0 are used along $y = y_0$, the stream function ψ_n, one grid point from the wall, is given by (22). On using (8) at an upper boundary, we obtain as conditions on the components of eigenvectors of M,

$$x_0 = 0$$

$$x_{n+1} = \frac{-3}{n+1} \sum_{j=1}^{n} j x_j - \frac{1}{2} x_n .$$

Case A. $a = 1$ and the θ_r are roots of

$$f(\theta)/(1 - \cos\theta) = 0$$

where

$$f(\theta) = (n+1)\sin(n-1)\,\theta - 6(n+1)\sin n\theta + 3(n-1)\sin(n+1)\,\theta + 2(n+1)\sin(n+2)\,\theta.$$

By considering $f(\theta)$ at points $r\pi/(n + \frac{1}{2})$ we deduce that there are at least $(n - 1)$ real roots in $(0, \pi)$. The remaining root is of the form $\theta_n = \pi + iz_n$ as may be seen by considering

$$F(z) \equiv (-1)^n \, if(\pi + iz).$$

$F(z)$ has two zeros apart from $z = 0$, as $F(\infty) < 0$ and $F'(0) > 0$. If one zero is at z_n the other is at $-z_n$ as $F(z)$ is an odd function. From (14) with $b = 0$ we deduce that the corresponding eigenvalue is

$$\lambda_n = -2\alpha(\operatorname{ch} z_n + 1). \tag{34}$$

This eigenvalue is negative and therefore (11) is stable. An eigenvalue of the form (34) does however give some difficulties when performing the time integration as will be shown in Section 6.3.

Case B. Numerical evidence suggests this case is stable.

Case C. Assuming n is odd analysis similar to that of Section 4.2 may be made and critical values of β/α are shown in Fig. 1.

6.2. *Semi-Infinite Region with Specified Velocity at Infinity*

We obtain similar results to those of the last section. In Case A there is again an eigenvalue of the form (34) and, for Case C, $(\beta/\alpha)_c \sim 2.186$ for large n.

6.3. *Explicit Time Integration Methods with Woods' Condition*

For Woods' condition with either type of condition at infinity, there will be difficulties in choosing a suitable stable method of replacing time derivatives in (11) even for the diffusion Case A. In the simple explicit method, an eigenvalue of the form (34) will give a restriction on the magnitude of the time step δt which is more severe than is usual. We require [10]

$$\delta t \leqslant \min_{1 \leqslant r \leqslant n} \left\{ \frac{-2 \operatorname{Rl}(\lambda_r)}{|\lambda_r|^2} \right\} = \frac{1}{\alpha(\operatorname{ch} z + 1)}$$

whereas the usual restriction is

$$\delta t \leqslant \frac{1}{2\alpha}$$

which applies if all θ_r are real. For example, if the stream function is specified at infinity, $n = 15$ and $V = 0$, there is an eigenvalue -4.951α when we require $\delta t < 0.404\alpha$ instead of the usual restriction $\delta t < 0.5\alpha$.

The DuFort–Frankel [11] method is always unstable when an eigenvalue of the form of (34) occurs. For (11) the method is

$$\boldsymbol{\zeta}^{s+1} - \boldsymbol{\zeta}^{s-1} = \delta t[(\mathbf{M} + 2\alpha\mathbf{I})\, \boldsymbol{\zeta}^{s} - \alpha\boldsymbol{\zeta}^{s+1} - \alpha\boldsymbol{\zeta}^{s-1} + \mathbf{b}^{s}],$$

where $\boldsymbol{\zeta}^s$ is the vorticity vector after s time steps and $\mathbf{b}^s$ is determined by the boundary conditions. We can write the method in the two stage form

$$\begin{bmatrix} \boldsymbol{\zeta}^{s+1} \\ \boldsymbol{\zeta}^{s} \end{bmatrix} = \mathbf{C} \begin{bmatrix} \boldsymbol{\zeta}^{s} \\ \boldsymbol{\zeta}^{s-1} \end{bmatrix} + \begin{bmatrix} \mathbf{b}^{s} \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{C}$ is the $2n \times 2n$ matrix

$$\mathbf{C} = \begin{bmatrix} (1 + \alpha \cdot \delta t)^{-1}\, \delta t(\mathbf{M} + 2\alpha \mathbf{I}) & (1 + \alpha \cdot \delta t)^{-1}\,(1 - \alpha \cdot \delta t)\, \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

This recurrence relation is unstable if any eigenvalue of $\mathbf{C}$ has absolute value greater than unity. The eigenvalues of $\mathbf{C}$ are

$$\frac{(\lambda_r + 2\alpha)\, \delta t \pm \sqrt{(\lambda_r + 2\alpha)^2 \cdot \delta t^2 + 4(1 + \alpha \cdot \delta t)(1 - \alpha \cdot \delta t)}}{2(1 + \alpha \cdot \delta t)} \qquad \text{for } r = 1, 2, \ldots, n.$$

One of these, when λ_r is given by (34), is

$$\frac{-\delta t \cdot \alpha \cdot \operatorname{ch} z_r - \sqrt{1 + \alpha^2 \cdot \delta t^2 \operatorname{sh}^2 z_r}}{1 + \alpha \cdot \delta t}$$

which is less than -1. This difficulty also occurs for the diffusion equation with derivative boundary conditions and is described in [12].

The effect of this instability is illustrated in an example in Section 7. No comparable difficulty arises with either Fromm's or Thom's condition. For the latter methods, we have shown that all eigenvalues are of the form (14) with θ_r real if there is no suction.

7. Numerical Examples

We consider, as a numerical example, flow along a wall $y = 0$ which has been brought impulsively to rest at time $t = 0$. We seek a solution $u(y, t)$ of the velocity equation,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} - V \frac{\partial u}{\partial y}, \tag{35}$$

where the flow is in the region $y > 0$ and $-V$ is the suction velocity at the wall. The boundary conditions are

$$\begin{aligned} &u(y, 0) = U, \qquad \text{for} \quad y \geqslant 0, \\ &u(0, t) = 0, \qquad u(\infty, t) = U, \qquad \text{for} \quad t > 0. \end{aligned} \tag{36}$$

U is the initial velocity of the fluid and the free stream velocity in the x-direction.

The solution may be found by integral transformation and on substituting it in

$$\zeta = -\frac{\partial u}{\partial y}$$

we obtain

$$\zeta(y, t) = -U\left[\frac{1}{\sqrt{\pi \nu t}} \exp\left\{-\left(\frac{y}{2\sqrt{\nu t}} - \frac{V}{2}\sqrt{\frac{t}{\nu}}\right)^2\right\}\right.$$
$$\left. - \frac{V}{2\nu} \exp\left(\frac{Vy}{\nu}\right) \operatorname{erfc}\left\{\frac{y}{2\sqrt{\nu t}} + \frac{V}{2}\sqrt{\frac{t}{\nu}}\right\}\right] \tag{37}$$

where erfc(x) is the complimentary error function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-x^2}\, dx.$$

In Figure 2 we show results obtained using the DuFort–Frankel method for all

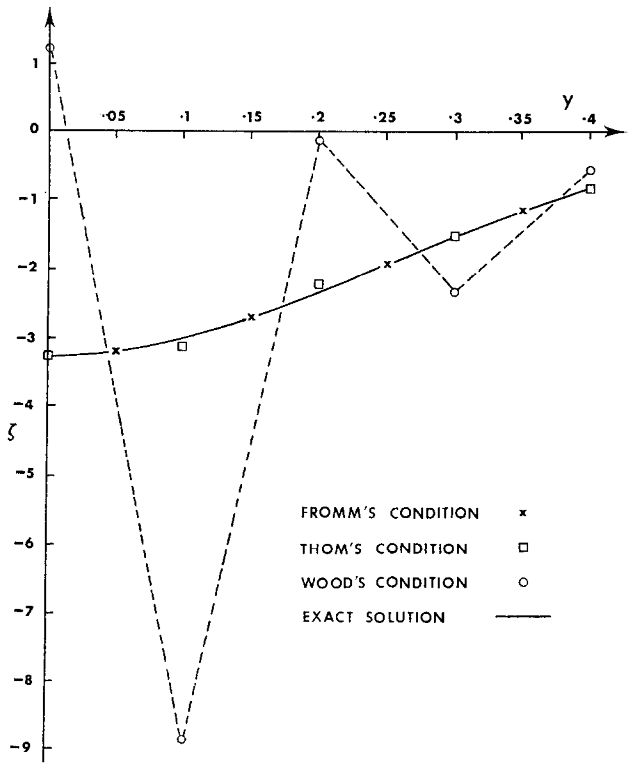

FIG. 2. $V = 0$ case showing instability for Wood's condition.

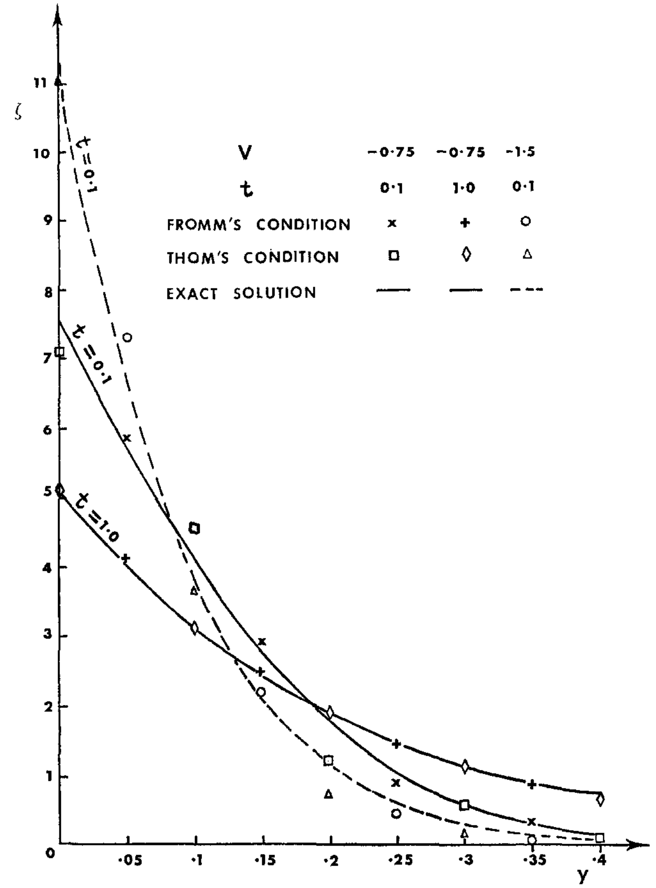

FIG. 3. Numerical examples of Thom's and Fromm's conditions.

three no-slip conditions with $V = 0$, when there is no suction. $h = 0.1$, $n = 15$, $\nu = 0.15$, $\delta t = 0.001$ and the results are for $t = 0.2$. In all cases the boundary "at infinity" is a line along which the velocity U is specified by $U = 1$. The results for Fromm's condition are displaced by a half mesh length as the wall is effectively along $y = -\frac{1}{2}h$. Instability can clearly be seen for the Woods' condition.

The results shown in Fig. 3 were obtained for Fromm's and Thom's conditions and the DuFort–Frankel method with $h = 0.1$, $n = 15$, $\delta t = 0.01$, $U = 1$, $\nu = 0.15$, and $V = -0.75$ and -1.5. (Thus $\alpha = 15$ and $\beta = -3.75$ and -7.5.) Fromm's condition gives more accurate results than Thom's, particularly for small values of t.

8. Restriction of Mesh Length

We have already seen that the ratio $\beta/\alpha = Vh/2\nu$ must be restricted if the numerical method is to be stable. This requirement is equivalent to a limitation on the size of h. As $t \to \infty$, the solution (37) tends to a steady-state solution whose form also indicates that h must be restricted. The steady-state solution is

$$\zeta(y, \infty) = \frac{UV}{\nu} e^{Vy/\nu}.$$

If h is too large, the vorticity will be concentrated into the region between the boundary and the first interior grid point and the finite-difference approximation will be completely inaccurate. The difference approximation

$$\frac{\zeta(h, \infty) - \zeta(0, \infty)}{h} = \frac{UV}{\nu}\left(\frac{e^{Vh/\nu} - 1}{h}\right)$$

to the derivative

$$\frac{\partial\zeta(\frac{1}{2}h, \infty)}{\partial y} = \frac{UV^2}{\nu^2} \cdot e^{Vh/2\nu}$$

will have relative error E, where

$$\begin{aligned} E &= \left|\left(1 - \left(\frac{\zeta(h, \infty) - \zeta(0, \infty)}{h}\right)\Big/\left(\frac{\partial\zeta(\frac{1}{2}h, \infty)}{\partial y}\right)\right|\\ &= \left| 1 - \frac{\nu}{Vh}\,(e^{Vh/2\nu} - e^{-Vh/2\nu})\right|\\ &= \left| 1 - \frac{\alpha}{\beta}\,\mathrm{sh}\left(\frac{\beta}{\alpha}\right)\right|, \end{aligned}$$

where $\beta/\alpha = -Vh/2\nu$. (The change of sign is made because the wall is a lower and not an upper boundary.) For example, if $\beta/\alpha = 1$, $E \simeq 0.18$ which represents quite a large error whereas for the second numerical example of Section 7, $\beta/\alpha = 0.25$ and 0.5 when $E \simeq 0.010$ and 0.042. It is hence necessary for the ratio β/α to be restricted and most problems would probably at least require $\beta/\alpha < 1$. This limitation on β/α will also apply to two-dimensional flows. The instabilities found in Sections 4–6 are not as likely to restrict the magnitude of β/α as the accuracy considerations presented here.

9. Conclusions

The main result obtained is that the no-slip conditions are stable if there is no suction at the wall. If there is a suction velocity V, the mesh size h must be chosen so that the ratio Vh/ν (where ν is the viscosity) is restricted in magnitude. As

Vh/ν is increased the finite-difference approximation becomes firstly inaccurate and secondly unstable. There is very little difference between the stability properties of Fromm's [1] and Thom's [4] conditions. The higher order Woods' [5] condition however does give difficulties when used with explicit time-integration methods. The type of analysis used can often be applied to other boundary conditions, however, it does not seem possible that it can be extended to deal directly with two-dimensional problems, because the coefficients of the differential equation will not be constant over the domain. Some two-dimensional problems (with constant coefficients) could be investigated by using the properties of direct products of matrices. The main value of the analysis as applied here is that it does make it possible to identify possible causes of instability.

References

1. J. E. Fromm, A method for computing nonsteady, incompressible viscous flows, Report No. LA-2910, Los Alamos Scientific Laboratory (1963).
2. J. E. Fromm, *Methods Comput. Phys.* **3** (1964), 345.
3. P. J. Taylor, The stability of boundary conditions in the numerical solution of the time-dependent Navier–Stokes equations, Current Paper No. 1066, Aeronautical Research Council (1970).
4. A. Thom, *Proc. Roy. Soc. A* **141** (1933), 651.
5. L. C. Woods, *Aeron. Quart.* **5** (1954), 176.
6. D. B. Russell, On obtaining solutions to the Navier–Stokes equations with automatic digital computers, Rep. & Mem. No. 3331, Aeronautical Research Council (1963).
7. A. Arakawa, *J. Comput. Phys.* **1** (1966), 119.
8. J. E. Fromm, Finite difference methods of solution of nonlinear flow processes with application to the Bénard Problem, Rep. No. LA-3522, Los Alamos Scientific Laboratory (1967).
9. R. Bellman, "Stability Theory of Differential Equations." McGraw-Hill, New York, 1958.
10. R. S. Varga, "Matrix Iterative Analysis." Prentice-Hall, Englewood Cliffs, N.J., 1962.
11. E. C. DuFort and S. P. Frankel, *Math. Tables Aids Comput.* **7** (1953), 135.
12. P. J. Taylor, *Comput. J.* **13** (1970), 92.